IIT GOA
INDIAN INSTITUTE OF TECHNOLOGY GOA
भारतीय प्रौद्योगिकी संस्थान गोवा

CHENNAI
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Edit distance of finite state transducers

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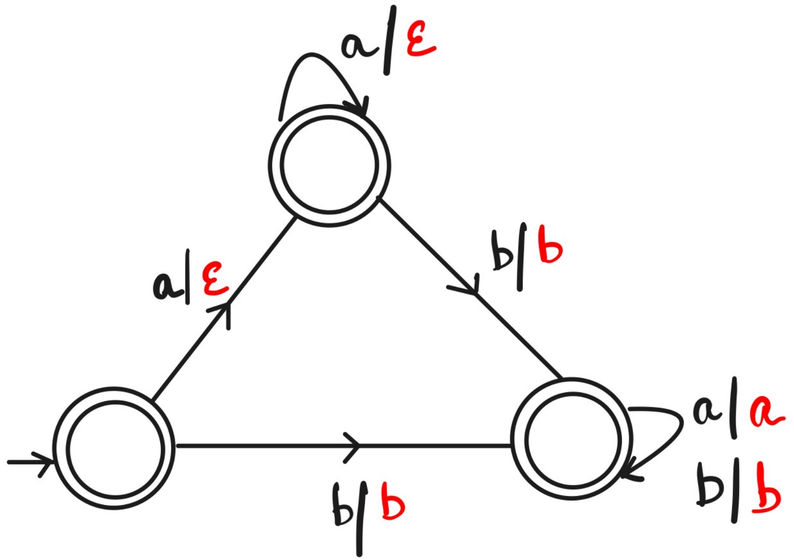
Joint work with **C. Aiswarya** (CMI) and **Amaldev Manuel** (IIT Goa)

Transducers

- Transducers are finite state automata that produce output words



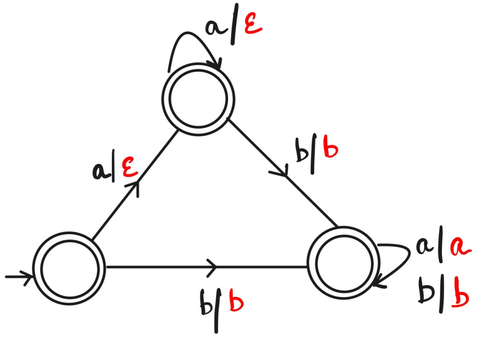
$aaba \rightarrow ba$

$bab \rightarrow bab$

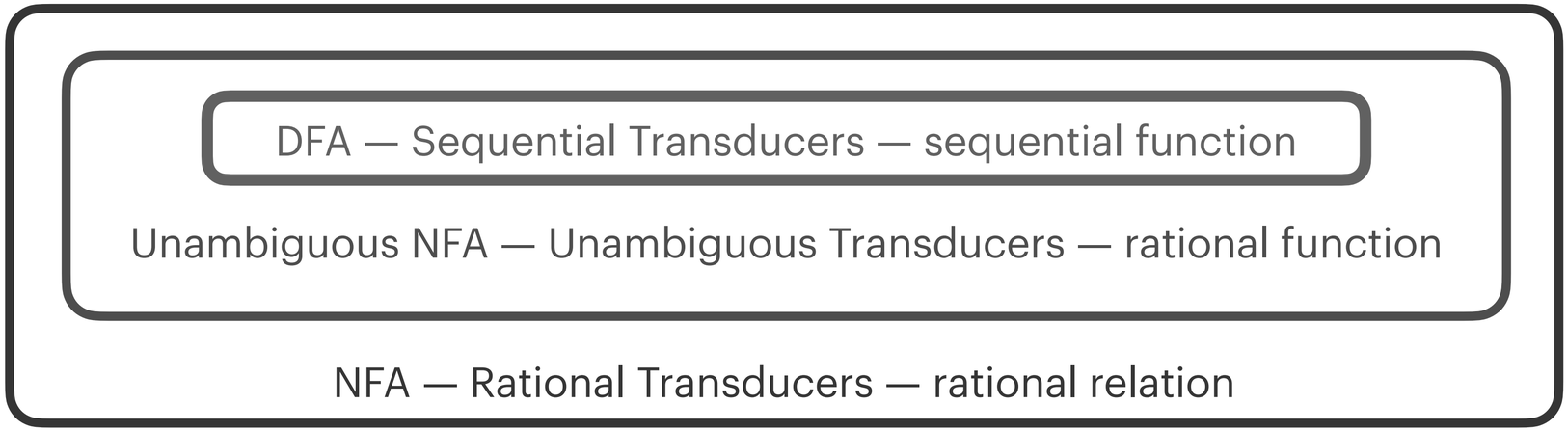
Delete all a's before first b

Transducers

- Transducers are finite state automata that produce output words



Delete all *a*'s before first *b*



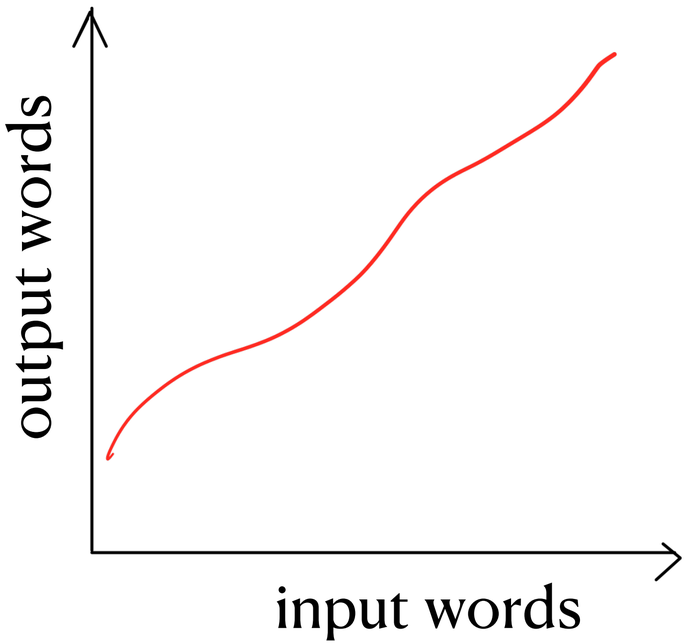
How do we compare transducers?

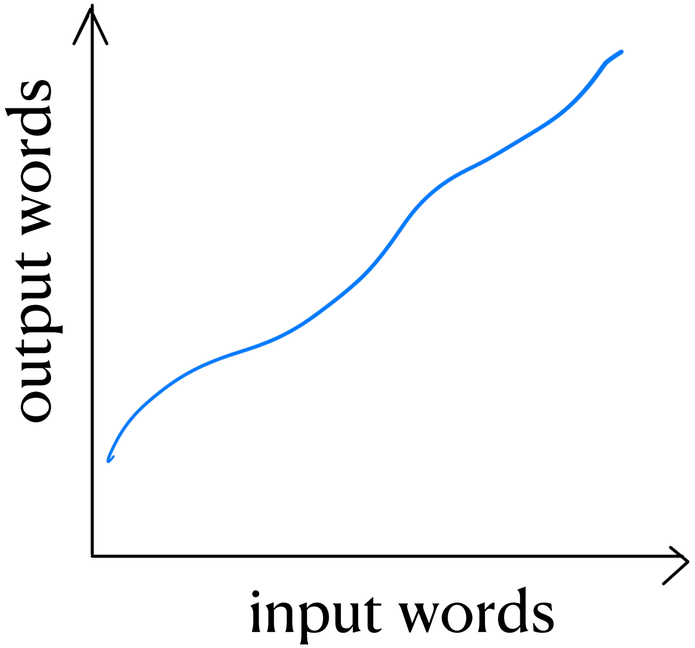
- Checking equivalence of two transducers
    - decidable for rational functions [Gurari-Ibarra' 1983],
    - decidable for regular functions [Gurari'1982,Culik-Karhumaki'1987]
    - open for polyregular functions [Bojanczyk'2018]
    - undecidable for rational relations [Griffiths'1968]

- Can we say something meaningful about non-equivalent transducers?

How do we compare transducers?

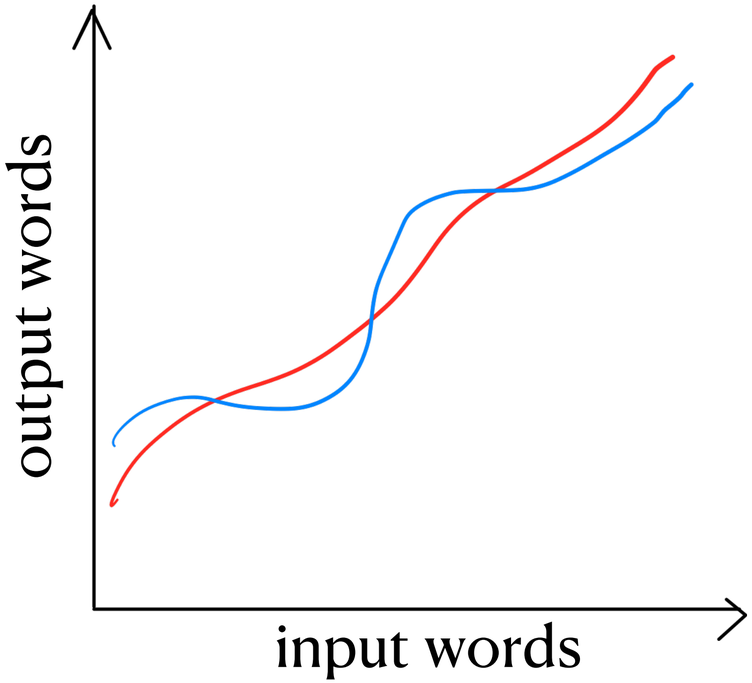
- Functional equivalence (on any input, the respective outputs are “exactly” the same)

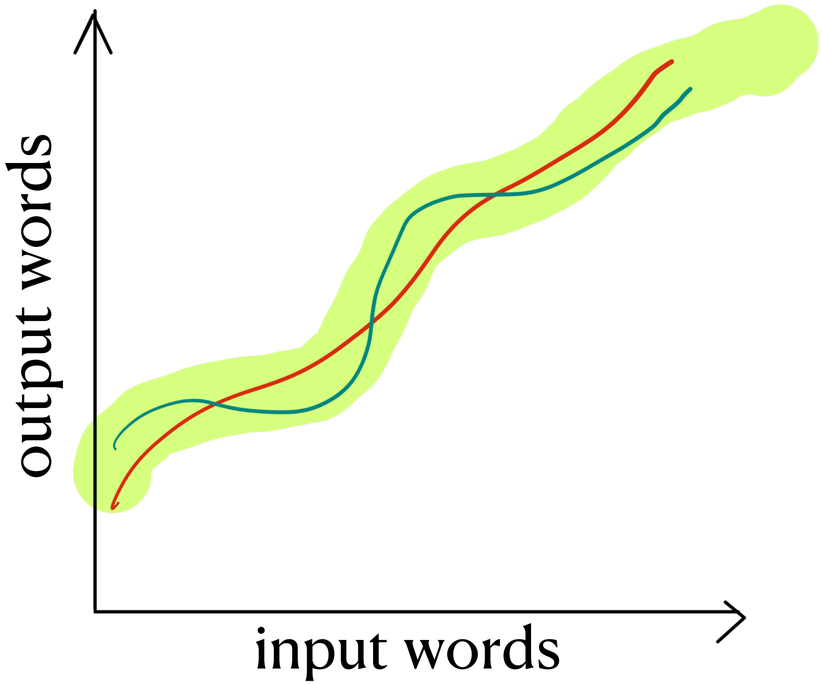




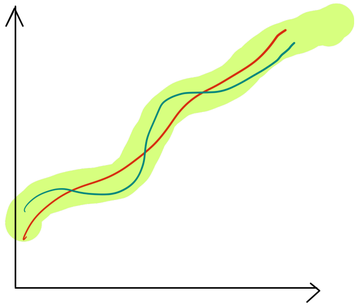
How do we compare transducers?

- Relax it : on any input, the respective outputs are close enough





Metric on transducers



- Let d be a metric on words. Lift it to word-to-word functions (transductions).

$$d(T_1, T_2) = \begin{cases} sup\ \{d(T_1(w), T_2(w)) \mid w \in dom(T_1)\} & \text{if } dom(T_1) = dom(T_2) \\ \infty & otherwise \end{cases}$$

- T_1 and T_2 are close if $d(T_1, T_2)$ is finite.

- Related work: adjacent functions [Reutenauer-Schützenberger ‘1991]

Edit Distances

- Given a set of edit operations,
    - Ex: insert a letter, delete a letter, or substitute a letter with another
- Edit distances between two words is the minimum number of edits required to convert one to another.

ababa

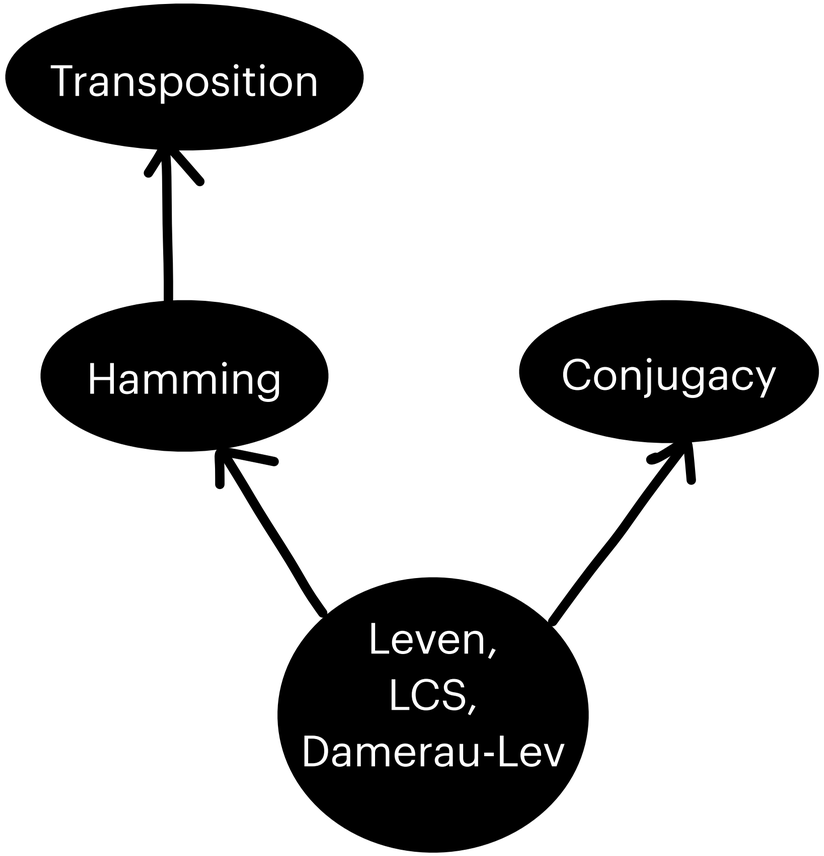
babab

Common Edit distances

Edit Distances	Edit operations
Hamming distance	letter-to-letter substitution
Transposition distance	swapping adjacent letters
Conjugacy distance	left and right cyclic shifts
Levenshtein edit distance	insertion, deletion, substitution
Longest common subsequence	insertion and deletion
Damerau-Levenshtein distance	Insertion,deletion,substitution and adjacent transposition

Edit distances - preorder relation

Edit Distances	Edit operations
Hamming distance	letter-to-letter substitution
Transposition distance	swapping adjacent letters
Conjugacy distance	left and right cyclic shifts
Levenshtein edit distance	insertion, deletion, substitution
Longest common subsequence	insertion and deletion
Damerau-Levenshtein distance	Insertion,deletion,substitution and adjacent transposition



Common Edit distances

Edit Distances	Edit operations
Hamming distance	letter-to-letter substitution
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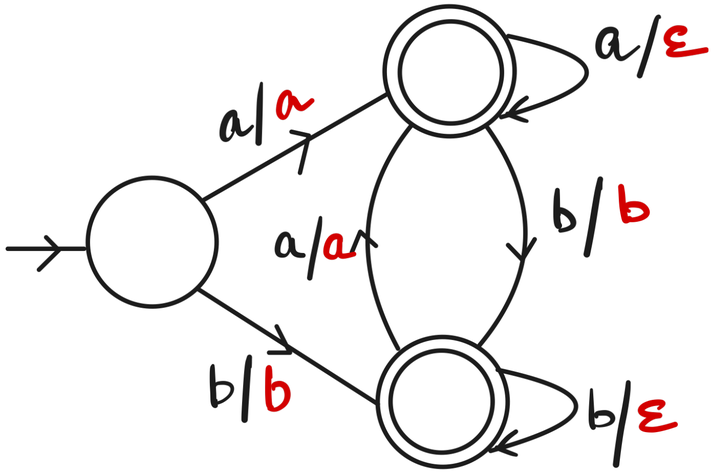
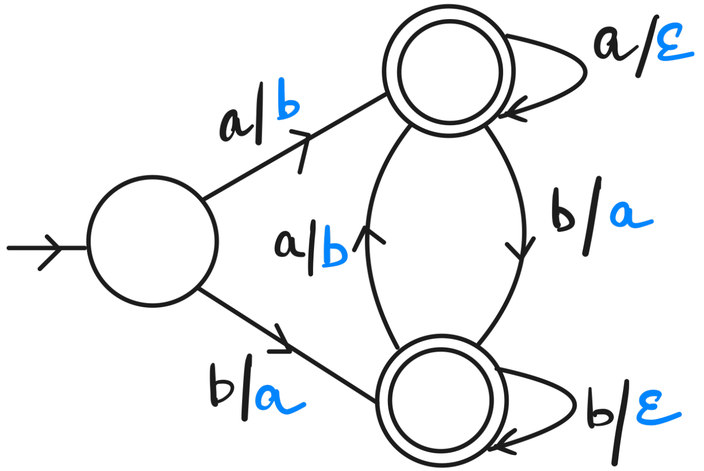
Conjugacy distance	left and right cyclic shifts
Levenshtein edit distance	insertion, deletion, substitution
Longest common subsequence	insertion and deletion
Damerau-Levenshtein distance	Insertion,deletion,substitution and adjacent transposition
Discrete	∅

Metric on transducers

Example



- For each block of a, output a
- For each block of b, output b



- For each block of a, output b
- For each block of b, output a

$$aaabbbabbba \to (ababa, babab)$$

- $d_{lev}(T_1, T_2) = 2$
- $d(T_1, T_2) = \infty$ if only
  - substitutions
  - cyclic shifts
  - adjacent swapping

Metric on transducers

Questions

$$d(T_1, T_2) = \begin{cases} sup\ \{d(T_1(w), T_2(w)) \mid w \in dom(T_1)\} & \text{if } dom(T_1) = dom(T_2) \\ \infty & otherwise \end{cases}$$

- Given T_1, T_2 is $d(T_1, T_2)$ computable? (Distance)
- Given T_1, T_2 is $d(T_1, T_2)$ finite? (Closeness)
- Given T_1, T_2 and $k \in \mathbb{N}$, is $d(T_1, T_2)$ at most k? (k-closeness)

Metric on transducers

Results

Problem	Input	Question
Distance Problem	transducers $\mathcal{T}_1, \mathcal{T}_2$	$d(\mathcal{T}_1, \mathcal{T}_2)$?
Closeness Problem	transducers $\mathcal{T}_1, \mathcal{T}_2$	Is $d(\mathcal{T}_1, \mathcal{T}_2) < \infty$?
k-closeness Problem	integer k, transducers $\mathcal{T}_1, \mathcal{T}_2$	Is $d(\mathcal{T}_1, \mathcal{T}_2) \leq k$?

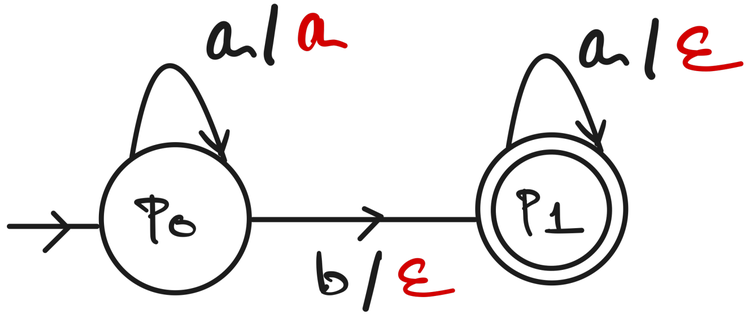
Proposition: Distance is computable iff closeness and k-closeness is decidable for integer-valued metrics

Theorem: Closeness and k-closeness for rational functions is decidable for all metrics $d \in \{d_{lev}, d_{lcs}, d_{damerau}, d_{conj}, d_{ham}, d_{trans}\}$.

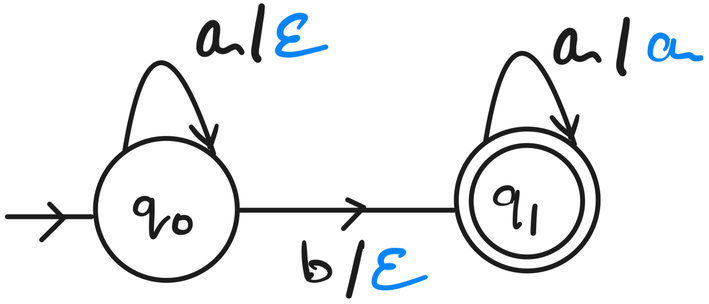
Closeness and k-closeness

- Given transducers T_1, T_2
    - Domain of T_1 and T_2 must be same.
    - Let T be the cartesian product of T_1 and T_2

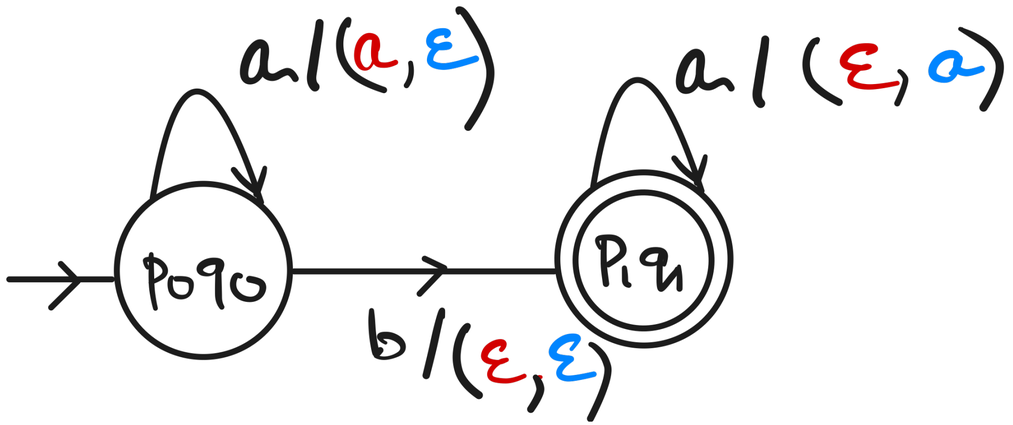
Cartesian product of two transducers



Output a's before b



Output a's after b



$$aaba \rightarrow (a, \epsilon) \cdot (a, \epsilon) \cdot (\epsilon, \epsilon) \cdot (\epsilon, a) = (aa, a)$$

Closeness and k-closeness

- Given transducers T_1, T_2
  - Domain of T_1 and T_2 must be same.
  - Let T be the cartesian product of T_1 and T_2

    — generates set of all pairs of output words of T_1, T_2 on any input
  - Loops of T - must generate output pairs of same length (Close w.r.t. d_{len})

k-closeness

For edit distances

- Given transducers T_1, T_2
  - Domain of T_1 and T_2 must be same.
  - Let T be the cartesian product of T_1 and T_2

    — generates set of all pairs of output words of T_1, T_2 on any input
  - Loops of T - must generate output pairs of same length (Close w.r.t. d_{len})

1. From T, construct an automaton that accepts w if $d(T_1(w), T_2(w)) \leq k$
2. Start with budget k. Non-deterministically do edits, update the budget and residues appropriately. Budget is not allowed to be negative.
3. Check if the language accepted is the domain of T. Yes: k-close; No: not k-close.

Closeness

- Given transducers T_1, T_2
  - Domain of T_1 and T_2 must be same.
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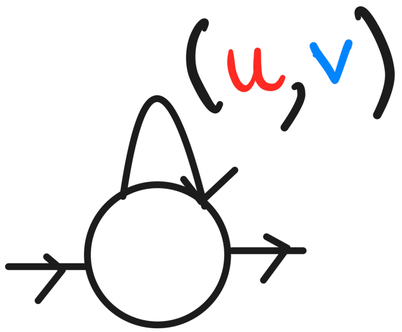
Identical output pairs?

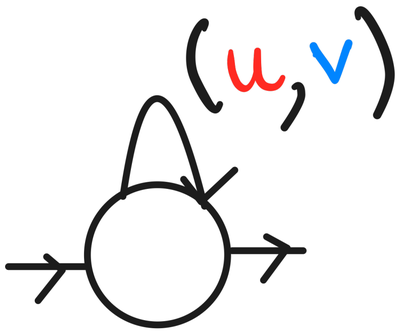
if $u = xy$ and $v = yx$

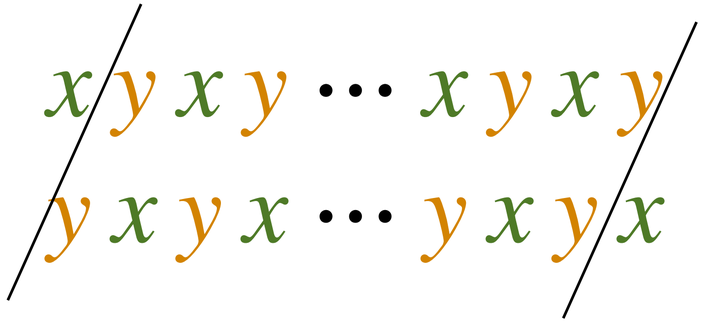


$$x\,y\,x\,y\cdots x\,y\,x\,y$$
$$y\,x\,y\,x\cdots y\,x\,y\,x$$

Conjugates

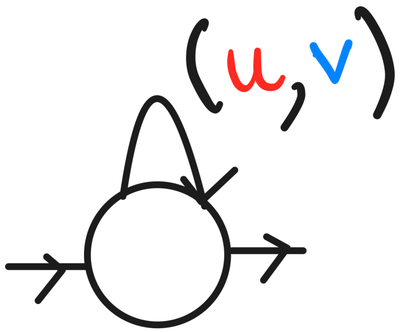
- u and v are conjugates if $u = xy$ and $v = yx$





Conjugates

- u and v are conjugates if $u = xy$ and $v = yx$
- Equivalently, if there exists a word z such that $uz = zv$ [Lyndon-Schützenberger'62]



$x\,y$

$y\,x$

Conjugates

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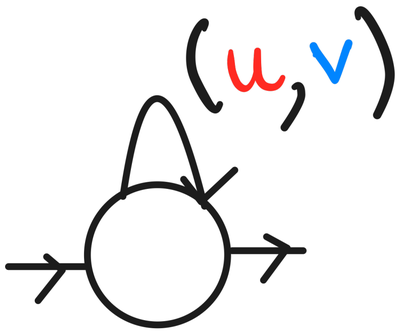

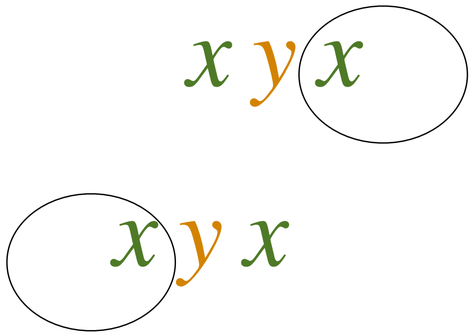

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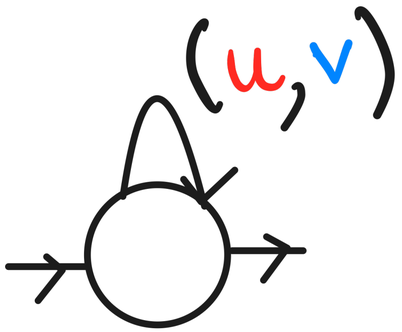
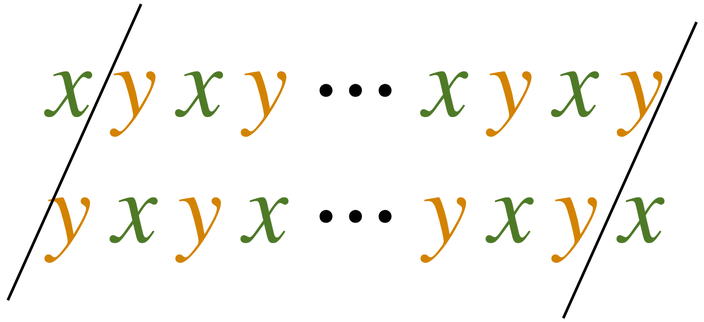


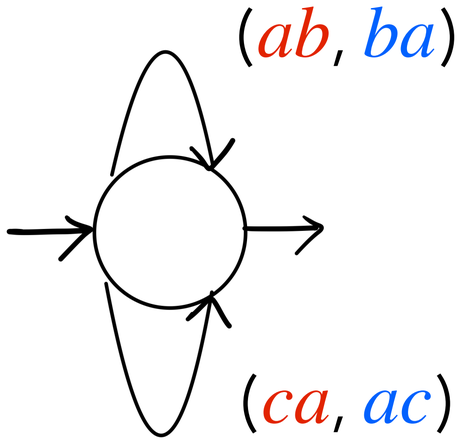


Closeness

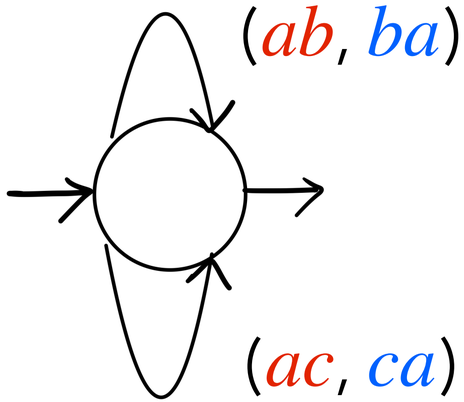
- Given transducers T_1, T_2
  - Domain of T_1 and T_2 must be same.
  - Let T be the cartesian product of T_1 and T_2
    — generates set of all pairs of output words of T_1, T_2 on any input
  - Loops of T - must generate output pairs of same length

Lemma: If T_1 and T_2 are close w.r.t metric $d \in \{d_{lev}, d_{lcs}, d_{damerau}, d_{conj}, d_{ham}, d_{trans}\}$, then every loop in T generates only conjugate pair of words..

More on conjugates



(*abca*, *baac*) is not conjugate



Any combination of pairs is conjugate

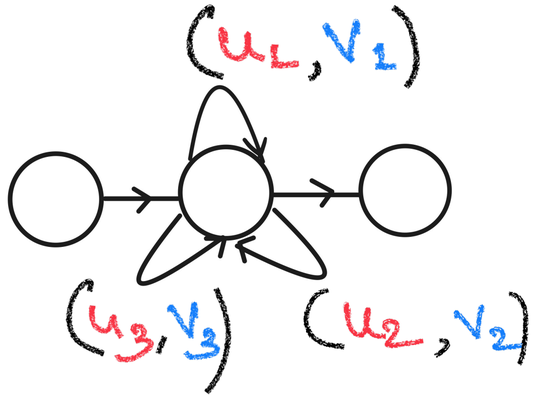
$$a\,b\cdots a\,b\,a\,c\cdots a\,c$$
$$b\,a\cdots b\,a\,c\,a\cdots c\,a$$

$$u\,z = z\,v$$

More on conjugates

G — set of pairs of words

G^* — consist of pairs obtained by point wise concatenation of some pairs in G



When is every pair in G^* conjugate?

Theorem[Aiswarya-Manuel-S. '2024]

The set G^* is conjugate iff G has a common witness

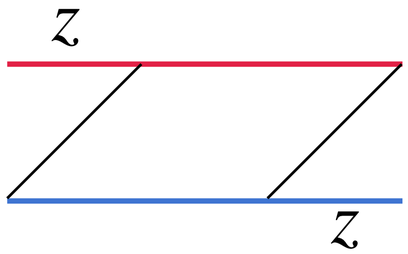
Closeness

For Levenshtein family

- Given transducers T_1, T_2
  - Domain of T_1 and T_2 must be same.
  - Let T be the cartesian product of T_1 and T_2

    — generates set of all pairs of output words of T_1, T_2 on any input

Lemma: T_1 and T_2 are close w.r.t. Levenshtein distance if and only if all the loops of T generate only conjugate words



More on conjugates

Theorem[Aiswarya-Manuel-S. '2024]

The set $(u_0, v_0)G_1^*(u_2, v_2)G_2^* \cdots G_k^*(u_k, v_k)$ is conjugate iff it has a common witness

Closeness

For Conjugacy

- Given transducers T_1, T_2
  - Domain of T_1 and T_2 must be same.
  - Let T be the cartesian product of T_1 and T_2

    — generates set of all pairs of output words of T_1, T_2 on any input

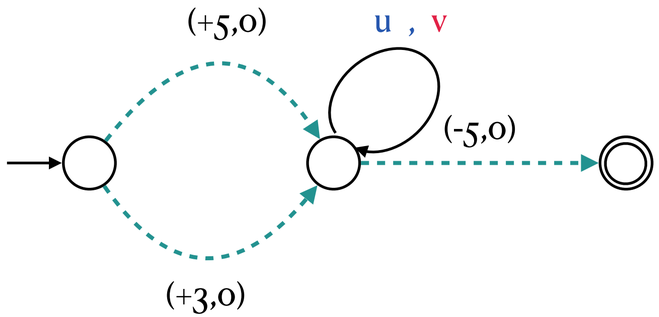
Proposition: T_1 and T_2 are close w.r.t. conjugacy distance if and only if T generate only conjugate words

- Conjugacy of a rational relation is decidable [Aiswarya-Manuel-S. '2024]

Closeness

For Hamming and transposition

- Output lengths must be equal for all words.
  - Loops generate words of same length
  - Delay between partial outputs depends only on the state.
- Check if all pairs in a loop (modulo the delay on border) are equal (up to some length depending on the delay).



Metric on transducers

Results

Problem	Input	Question
Distance Problem	transducers $\mathcal{T}_1, \mathcal{T}_2$	$d(\mathcal{T}_1, \mathcal{T}_2)$?
Closeness Problem	transducers $\mathcal{T}_1, \mathcal{T}_2$	Is $d(\mathcal{T}_1, \mathcal{T}_2) < \infty$?
k-closeness Problem	integer k, transducers $\mathcal{T}_1, \mathcal{T}_2$	Is $d(\mathcal{T}_1, \mathcal{T}_2) \leq k$?

Proposition: Distance is computable iff closeness and k-closeness is decidable for integer-valued metrics

Theorem: Closeness and k-closeness for rational function is decidable for all metrics $d \in \{d_{lev}, d_{lcs}, d_{damerau}, d_{conj}, d_{ham}, d_{trans}\}$.

Related notions and generalisations

Diameter of a Rational Relation

- The diameter of a rational relation R w.r.t. a metric d is the supremum of distance of each pair of words in R

$$dia_d(R) = \sup\{d(u, v) \mid (u, v) \in R\}$$

- Related Work: rational relation with bounded delay [Frougny-Sakarovitch'1991]

Diameter of a Rational Relation

Questions

Problem	Input	Question
Diameter Problem	rational relation R	$dia_d(R)$?
Bounded Diameter Problem	rational relation R	Is $dia_d(R) < \infty$?
k-bounded Diameter Problem	integer k, rational relation R	Is $dia_d(R) \leq k$?

Diameter of a Rational Relation

Results

Problem	Input	Question
Diameter Problem	rational relation R	$dia_d(R)$?
Bounded Diameter Problem	rational relation R	Is $dia_d(R) < \infty$?
k-bounded Diameter Problem	integer k, rational relation R	Is $dia_d(R) \leq k$?

Proposition: Diameter problem of a rational relation is mutually reducible to distance problem of two rational functions

Proposition: Diameter problem of a rational relation is mutually reducible to distance problem of two rational functions

- Distance -> Diameter
  - Given two transducers T_1, T_2, check if their domains are equal
  - $d(T_1, T_2) = dia_d(R)$ where R is the relation generated by cartesian product of T_1 and T_2
- Diameter -> Distance
  - By virtue of [Nivat'1968] theorem

Diameter of a Rational Relation

Results

Problem	Input	Question
Diameter Problem	rational relation R	$dia_d(R)$?
Bounded Diameter Problem	rational relation R	Is $dia_d(R) < \infty$?
k-bounded Diameter Problem	integer k, rational relation R	Is $dia_d(R) \leq k$?

Proposition: Diameter problem of a rational relation is mutually reducible to distance problem of two rational functions

Corollary: All the above problems are decidable for rational relation w.r.t. metrics $d \in \{d_{lev}, d_{lcs}, d_{damerau}, d_{conj}, d_{ham}, d_{trans}\}$

Index of relation in a composition closure

- Index of a rational relation R in the composition closure of S is the smallest integer k such that R is contained in at most k-fold composition of S

$$R \subseteq \bigcup_{0 \leq i \leq k} \underbrace{S \circ S \cdots \circ S}_{i \text{ times}}$$

- Example:

  $\{a, b\}^* \times \{a, b\}^*$

  - S — deletes the first a if exists on any input
  - R_k — deletes first k a's if exist on any input
  - R — delete all a's on any input

  - $Index(R_k, S) = k$
  - $Index(R, S) = \infty$

Index of relation in a composition closure

Questions

Problem	Input	Question
Index Problem	rational relation R, S	$\text{Index}(R, S)$?
Bounded (or Finite) Index Problem	rational relation R, S	Is $\text{Index}(R, S) < \infty$?
k-bounded Index Problem	integer k, rational relation R, S	Is $\text{Index}(R, S) \leq k$?

Index of relation in a composition closure

Results

Problem	Input	Question
Index Problem	rational relation R, S	$\text{Index}(R, S)$?
Bounded (or Finite) Index Problem	rational relation R, S	Is $\text{Index}(R, S) < \infty$?
k-bounded Index Problem	integer k, rational relation R, S	Is $\text{Index}(R, S) \leq k$?

Lemma: It is undecidable to check if a rational relation has a bounded index in the composition closure of an arbitrary rational relation

Metrizable Relation

Proposition: The index of a rational relation in the composition closure of a d-metrizable relation is computable for $d \in \{d_{len}, d_{lev}, d_{lcs}, d_{dl}, d_h, d_{trans}, d_{conj}\}$

- Graph of a relation S - vertices (words) , edge (between related words in S)
- $d_S(u, v)$ = length of the shortest path between u and v in the graph of S
- S is d- metrizable if d_S is equivalent to metric d up to boundedness.

Index of relation in a composition closure

Results

Problem	Input	Question
Index Problem	rational relation R, S	Index(R, S)?
Bounded (or Finite) Index Problem	rational relation R, S	Is Index$(R, S) < \infty$?
k-bounded Index Problem	integer k, rational relation R, S	Is Index$(R, S) \leq k$?

Lemma: It is undecidable to check if a rational relation has a bounded index in the composition closure of an arbitrary rational relation

Corollary: All the above problems are decidable for rational relation in the composition closure of d- metrizable relation for

$$d \in \{d_{len}, d_{lev}, d_{lcs}, d_{dl}, d_h, d_{trans}, d_{conj}\}$$

Conclusion

- We have defined the following notions
    - Distance between rational functions
    - Diameter of rational relation
    - Index of a rational relation in a composition closure
- All are computable w.r.t. metrics $d \in \{d_{len}, d_{lev}, d_{lcs}, d_{dl}, d_{h}, d_{trans}, d_{conj}\}$

Thank you

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